DOCUMENT RESUME

ED 389 574 SE 057 217

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TITLE	Middle School Students' Understanding of Average: A Problem-Solving Approach.
SPONS AGENCY	Ford Foundation, New York, N.Y. Education and Research Div.
PUB DATE	Oct 95
NOTE	8p.; Paper presented at the Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education (17th, Columbus, OH, October 21-24, 1995). For entire conference proceedings, see SE 057 177.
PUB TYPE	Reports - Research/Technical (143) -- Speeches/Conference Papers (150)
EDRS PRICE	MF01/PC01 Plus Postage.
DESCRIPTORS	*Cognitive Processes; *Concept Formation; Grade 6; Grade 7; Grade 8; Intermediate Grades; Junior High Schools; *Junior High School Students; *Mathematics Instruction; Middle Schools; *Statistics; Teaching Methods
IDENTIFIERS	*Arithmetic Mean; *Middle School Students

ABSTRACT

This study used an open-ended problem-solving approach to teaching and assessing middle school students' understanding of the concept of arithmetic average. Three main results of this study show evidence of positive instructional impact on students' understanding of the concept of average: (1) the number of students who gave correct answers increased from pretest to posttest; (2) on the posttest, more students used appropriate strategies to solve the average problems than on the pretest; and (3) more students used multiple representations on the posttest to explain their solutions than on the pretest. The findings of this study indicate that learning the concept of average is cognitively more complex than the computational algorithm suggests. However, with appropriate instruction, students can have an understanding of the concept beyond the computational algorithm. (Author)

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Paper presented at the Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education

(17th PME-NA, Columbus, OH, October 21-24, 1995)

MIDDLE SCHOOL STUDENTS' UNDERSTANDING OF AVERAGE: A PROBLEM-SOLVING APPROACH

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This study used an open-ended problem-solving approach to teaching and assessing middle school students' understanding of the concept of arithmetic average. Three main results of this study show evidence of positive instructional impact on students' understanding of the concept of average: (1) the number of students who gave correct answers increased from pretest to posttest; (2) on the posttest, more students used appropriate strategies to solve the average problems than on the pretest; (3) more students used multiple representations on the posttest to explain their solutions than on the pretest. The findings of this study indicate that learning the concept of average is cognitively more complex than the computational algorithm suggests. However, with appropriate instruction, students can have an understanding of the concept beyond the computational algorithm.

Arithmetic average is one of the important and basic concepts in data analysis and decision making. It is not only an important concept in statistics, but also an everyday-based concept (National Council of Teachers of Mathematics (NCTM), 1989). The arithmetic average is found by adding the values to be averaged and dividing the sum by the number of values that were summed. Although the computational algorithm suggests that arithmetic average is a simple concept to understand, previous research (e.g., Cai, 1995; Mevarech, 1983; Pollatsek, Lima, & Well, 1981; Strauss & Bichler, 1988) has indicated that both pre-college and college students have many misconceptions about the average concept. The misconceptions are not due to students' lack of the procedure for calculating an average, rather they are due to their lack of understanding of the concept of average.

The purpose of this study was to examine students' existing understanding of the average concept as well as the impact of open-ended problem solving instruction on their understanding of the concept. This study is an extension of an earlier study in which Cai (1995) used a multiple-choice task and an open-ended task to examine sixth-grade students' knowledge of arithmetic average. He performed a fine-grained cognitive analysis of the students' written responses. He found that the majority of the students knew the "add-them-all-up-and-divide" algorithm for calculating average, but only about half of the students showed evidence of having an understanding of the concept of average. The earlier study (Cai, 1995) also suggests the value of using an open-ended task to assess students' understanding of the average concept and to examine their problem-solving processes. This study extended the earlier study in two ways: (1) this study used two open-ended tasks to examine middle school students' knowledge of arithmetic average; and (2) this study also examined the instructional impact on students' understanding of the arithmetic average through a pretest and posttest design.

Preparation of this paper was supported in part by a grant from the Ford Foundation. Any opinions expressed herein are those of the authors and do not necessarily represent the views of the Ford Foundation.

Method

Subjects

Subjects numbered about 150 middle school students from a public school in a large urban school district. Students in the school are ethnically and culturally diverse, and 75% of the students are on a free or reduced lunch prcgram. In this paper, only those students who took both the pretest and the posttest are used in the analysis, which includes 123 students (46 sixth-graders, 33 seventh-graders, and 44 eighth-graders). It should be indicated that students had been briefly exposed to the average concept in previous years.

Pretests and Posttests

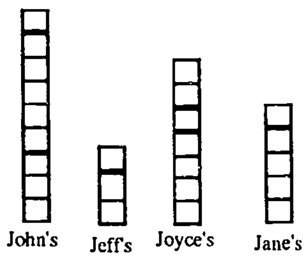
Figure 1 shows the two tasks used as pretests and posttests. In these tasks, students were asked to provide answers and, importantly, they were also asked to explain how they found their answers. In particular, Problem 1 requires students to figure out a simple mean of four numbers, and Problem 2 requires students to find a missing number when the first four numbers and the average of the five numbers (including the missing number) are presented graphically. In order to solve Problem 2, students must have a well-developed understanding of the average concept. Students were allowed about 15 minutes to complete these two problems. The posttest, which consisted of the same two problems as the pretest, was given about six months after the pretest.

Instructional Treatment

In this study, teachers used an open-ended problem-solving approach to teach the average concept with understanding. The instructional materials included those developed by Bennett, Maier, & Nelson (1988), which emphasize "averaging" as

Problem 1

John, Jeff, Joyce, and Jane each has a stack of blocks, which are shown below.

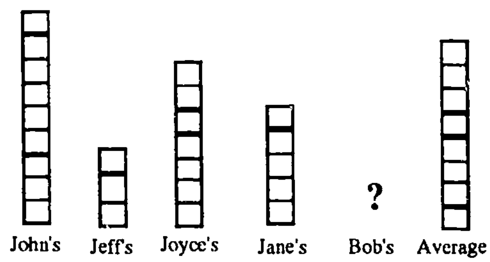


What is the average number of blocks for those four people?

Answer:
Explain how you found your answer.

Problem 2

Later Bob joined them. When Bob came in, the average number of blocks for John, Jeff, Joyce, Jane,



and Bob became 8.
How many blocks did Bob have so that the average for the five people was 8?

Answer:
Explain how you found your answer.

Figure 1. Tasks

an evening-off process. The materials stress that averaging can be used as an effective tool for making sense of a set of data rather than as a simple computation process. In addition to using the materials developed by Bennett et al. (1988), teachers also used a variety of average-related problems in their classroom (Meyer, Browning, & Channell, 1995). The teachers met with two university professors (the authors) regularly to discuss instructional materials and approaches. The teachers were encouraged to develop their own instructional materials based on the discussions in the regular meetings. The focus of the discussions was on ways of teaching the average concept with understanding, not just on the computational algorithm.

Data Coding and Analysis

Data coding and analysis were completed using a classification scheme adapted from Cai (1995). In particular, each response was coded with respect to four distinct perspectives: (1) numerical answer, (2) mathematical error, (3) solution strategy, and (4) representation. To ensure the inter-rater reliability, the two authors randomly selected 20% of the student responses and coded them independently. The inter-rater agreement ranged from 87% to 99%.

Results

Since grade level differences were not a focus of this study, the results are reported in an aggregated manner. There are three separate sections.

Numerical Answer and Mathematical Error

The numerical answer was what the student provided on the answer space on each task, and was judged correct or incorrect. With respect to the correctness of numerical answers, students improved significantly from the pretest to the posttest. Specifically, on the pretest, only 51 and 19 students respectively answered Problems 1 and 2 correctly. On the posttest, however, 104 and 84 students respectively gave the correct answers for Problems 1 and 2. Examination of the correctness of both problems shows that the percentages of students who gave correct answers for both problems increased significantly from 11% (13 of 123) on the pretest to 64% (79 of 123) on the posttest ($z = 7.57$, $p < .001$). The significant increase in students with correct answers from the pretest to posttest provides evidence of the instructional impact on student understanding of the average concept.

Examination of paired answers on the pretest shows that 80% (41 of 51) of the students who were able to solve Problem 1 failed to correctly solve Problem 2. On the posttest, 24% (25 of 104) of the students who were able to solve Problem 1 were still unable to correctly solve Problem 2, but the percentage is statistically smaller than on the pretest ($z = 6.67$, $p < .001$). This implies that after instruction students had a better understanding of the average concept. Interestingly, a few students correctly solved Problem 2 without also correctly solving Problem 1.

Fewer students made mathematical errors on the posttest than on the pretest. However, error analysis shows that students who did not correctly solve the problems tended to make similar types of errors on both tests. For example, a common error that students made in solving Problem 2 was to incorrectly apply the computational algorithm. For example, some students added the numbers of John's blocks (9), Jeff's (3), Joyce's (7), Jane's (5), and the average (8), got a sum of 32, then divided the sum by 5. The students typically gave the whole number part of the quotient (6) as the answer. These students appeared to know the computational procedure for calculating an average (i.e., "add-them-all-up-and-divide"), but they appeared to not know what should be added, what should be divided, or divided by. Thus, although student performance in solving the average problems improved significantly from pretest to posttest, a small proportion of the students still showed a lack of conceptual understanding of the arithmetic average.

Solution Strategy

Three solution strategies were identified, which are described in Table 1. On the pretest, only 42 and 17 students respectively gave a clear indication of using one of the three identified strategies in solving Problems 1 and 2. On the posttest, 94 and 66 students respectively gave a clear indication of using one of the three identified strategies in solving Problems 1 and 2.

Moreover, on the posttest, nearly 50% of the students gave clear indications of using solution strategies in solving both problems, but only 11% of them did so in the pretest. The difference between use of strategies on the pretest and posttest is statistically significant ($z = 6.26$, $p < .001$). This significant increase in the number of students who gave clear indications of using identified solution strategies from the pretest to posttest provides further evidence that instruction had a positive impact on student understanding of the average concept.

On the pretest, students most frequently used the average formula to solve the problems. On the posttest, the number of students who used average formula increased, but the increase was not as dramatic as that for leveling strategy. In fact, only a few students used the leveling strategy on the pretest, but over 40 students used the leveling strategy on the posttest. It should be noted that for those students who gave clear indications of using identified solution strategies in Problems 1 and 2, the majority of them (77%) tended to use the same solution strategies on both problems, either on the pretest or on the posttest. For example, if a student used the leveling strategy to solve Problem 1, he/she would most likely use the same strategy to solve Problem 2.

Representations

The representations were classified into the following categories: verbal (written words), symbolic (mathematical expressions), pictorial (drawings), and any combination of these three. Table 2 shows the number of students who used various representations.

From pretest to posttest, the number of students who did not provide explanations of their solutions decreased. In particular, on the pretest 14 and 29 students respectively did not provide an explanation in solving Problems 1 and 2; while on the posttest, only 2 and 12 students respectively did not provide explanations for Problems 1 and 2. Not only did more students provide explanations on the posttest

Table 1. *Descriptions of Solution Strategies and Frequency of Students Using Each of Them*

Strategy / Description	Number of Students: Pretest P1	Pretest P2	Posttest P1	Posttest P2
Strategy 1 (Using Average Formula): The student used the average formula to solve the problems. For example, in solving the first problem, students added blocks that John, Jeff, Joyce, and Jane have, then divided the sum by four. In solving the second problem, students multiplied the 5 by 8, got 40, then subtracted the number of blocks that John, Jeff, Joyce, and Jane had, so the answer was 16 [i.e., $8 \times 5 - (9 + 3 + 7 + 5) = 16$].	39	15	50	24
Strategy 2 (Leveling): Students tried to even-off the blocks to get the average number of blocks for John, Jeff, Joyce, and Jane in solving the first problem. In the second problem, students tried to use the average number of blocks as the leveling base, then found the number of blocks Bob had.	3	2	44	40
Strategy 3 (Guess-and-Check): The student first chose a number for Bob, then checked to see if the average was 8. If the average was not 8, then he/she chose another number for Bob and checked again, until the average was 8.	0	0	0	2
Total	42	17	94	66

than on the pretest, but also the quality of student explanations improved from pretest to posttest. For example, more students on the posttest tended to use multiple representations (i.e., any combination of verbal, pictorial, and symbolic representations) to explain their solution processes. In fact, only about 10% of the students used multiple representations on the pretest; while about 40% of the students used multiple representations on the posttest.

The representations students used appear to be related to the strategies they employed. For example, when students used the average formula to solve the problems, they tended to use symbolic-related representations in their explanations. While when students used leveling strategies, they tended to use pictorial-related representations in their explanations.

Discussion

This study used a problem-solving approach to teaching and assessing middle school students' understanding of the concept of arithmetic average. The results of this study suggest that for the pretest a majority of the students only knew the "add-them-all-up-and-divide" algorithm of calculating average. On the posttest, however, the number of students with conceptual understanding increased dramatically. The findings of this study provide evidence of positive instructional impact on students' understanding of the average concept. This evidence includes: (1) the number of students with correct answers increased from pretest to posttest;

Table 2. *Frequency of Students Using Various Representations in Pretest and Posttest*

	Number of Students			
	Pretest		Posttest	
	P1	P2	P1	P2
Verbal	74	60	46	42
Pictorial	3	6	11	10
Symbolic	19	14	11	10
Combination	13	14	53	49
Without Explanation	14	29	2	12

(2) more students on posttest than on pretest gave a clear indication of using appropriate strategies; (3) not only did more students provide explanations on the posttest than on the pretest, but also more students used multiple representations to explain their solutions.

The results of this study provide further evidence that learning the concept of average is cognitively more complex than the computational algorithm suggests, as was shown in previous studies (e.g., Cai, 1995; Strauss & Bichler, 1988). This study shows that if appropriate instructional approach and materials are used in the classroom, students will have an understanding of the average concept, not just the computational algorithm. This study also shows the appropriateness of using open-ended problems to teach and assess students' conceptual understanding of the arithmetic average.

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